University of Manchester Spring 2010

The Standard Model of Particle Physics Exercises I

to be handed in by 29 March 2010

1 The Cross Section for $e^+e^- \to \mu^+\mu^-$ **12 points**

Calculate the total cross section and the angular distribution $d\sigma/d\Omega$ in the c.m. system for the reaction

$$e^-(p_1)\, e^+(p_2) \to \mu^-(p_3)\, \mu^+(p_4)$$

in lowest order QED. For $s = (p_1 + p_2)^2 \gg m_e^2, m_\mu^2$ the electron and muon masses can be set to zero.

Perform the calculation in the way as presented in the lecture but work out all intermediate steps explicitely. The necessary formulae you find in the slides for the lecture `http://www-d0.fnal.gov/~schwanen/lecture_sm_10.pdf`. Take your time and try to understand the formulae you use.

2 Maxwell's equations **3 points**

Derive Maxwell's equations from the Lagrange density

$$\mathcal{L}(x) = -\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) + eA_\mu(x)\bar{\Psi}(x)\gamma^\mu\Psi(x)$$

using Hamilton's principle.

3 The QCD Lagrangian **7 points**

(a) Show that

$$-\frac{1}{4}\, G^a_{\lambda\rho}(x)\, G^{a\,\lambda\rho}(x) = -\frac{1}{2}\,\mathrm{Tr}(G_{\lambda\rho}(x)\, G^{\lambda\rho}(x))$$

where $G_{\lambda\rho}(x)$ is the field strength matrix for the gluons.

(b) Show that a gauge transformation for the quarks and the gluon potentials as given in the lecture leads to

$$G_{\lambda\rho}(x) \to U(x)\, G_{\lambda\rho}(x)\, U^\dagger(x)\ ,$$

$$D_\lambda\, q(x) \to U(x)\, D_\lambda\, q(x)\ ,$$

$$\mathcal{L}_{\mathrm{QCD}}(x) \to \mathcal{L}_{\mathrm{QCD}}(x)\ ,$$

where $\mathcal{L}_{\mathrm{QCD}}(x)$ is the QCD Lagrangian as given in the lecture. Calculate all intermediate steps yourself including those already outlined in the lecture.

4 Gauge Theories 18 points

Consider a theory with the Lagrange density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2\, A_\mu A^\mu.$$

(a) Derive the wave equation for the field A_μ and hence, or otherwise, justify calling m the mass of the particles associated with the field.

(b) Show that this Lagrangian is not invariant under U(1) transformations of the field.

The generalization to non-Abelian fields leads to

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + m^2\mathrm{Tr}(A_\mu A^\mu),$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and $A_\mu \equiv T^a A^a_\mu$ (T_a are the generators of SU(N) transformations in the fundamental representation).

In fact, it is possible to permit massive vector fields without spoiling gauge invariance and without introducing any new elementary particles. The aim of the rest of the question is to work out how this comes about.

(c) First introduce the SU(3) valued fields $V(x)_{ij}$ (i.e. V is an N $\times$ N matrix for an SU(N) gauge group) and postulate that, under gauge transformations (and suppressing indices),

$$V \to UV.$$

Now define the object

$$C_\mu \equiv -\frac{i}{g}V(x)\partial_\mu V^{-1}(x).$$

How does the combination $A_\mu - C_\mu$ transform under gauge transformations? Hence construct a gauge invariant mass term for the gauge field A_μ. What other physics do you appear to have introduced?

In fact the additional fields which were introduced do not actually have any physics content, so there is no "other physics" to worry about.